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Thermal transient behaviour of a bi-layer material with a non-plane interface

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Abstract

In this study, we propose studying the behaviour in a transitory regime of a bi-layer material with a non-plane interface. The aim of this paper is to show if a "*thermal diode*" effect in transient regime can exist, knowing that there are no thermal diode effects in steady-state regime. The question is to know if the response of a system can be modified if the stimulation is changed with the detection at the boundaries of the system. For instance, in the case of a "Flash" experiment where the stimulation represents a heat flux and the detection corresponds to a temperature measurement.

Keywords: Bi-layer material; Non-plane interface; Transient regime

1. Introduction

In the case of a homogeneous material subjected to a uniform heat flux stimulation in the front face (Fig. 1), the back face temperature response can be easily obtained using the thermal quadrupoles technique [1]:

$$\begin{bmatrix} \theta_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \theta_o \\ \phi_o \end{bmatrix} \tag{1}$$

θ_i and θ_o being the Laplace transforms of the front (input: i) and rear (output: o) faces temperatures respectively. ϕ_i and ϕ_o denote the Laplace transforms of the front and rear faces fluxes and A, B, C, and D represent the coefficients of the inverse transfer matrix of the system:

$$A = D = \mathrm{ch}(\alpha e) \tag{2}$$

$$B = \frac{1}{\lambda \alpha S} \mathrm{sh}(\alpha e) \tag{3}$$

$$C = \lambda \alpha S \, \mathrm{sh}(\alpha e) \tag{4}$$

with: $\alpha^2 = p/a$

In the case of an insulated system ($\phi_o = 0$):

$$\theta_o = \frac{\phi_i}{C} \tag{5}$$

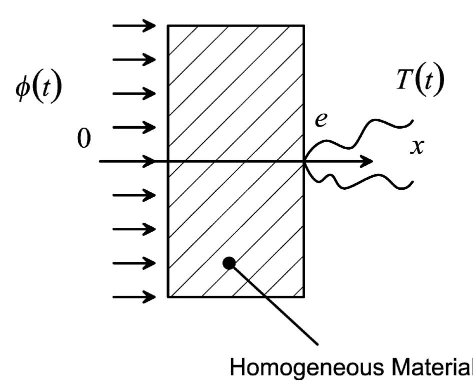

Fig. 1. Homogeneous material.

The impedance of the system is given by: $\frac{1}{C}$.

A homogeneous material is symmetric and thus reversible. So, if the inlet and outlet of the system are swapped, the response is the same.

Let us imagine now a non-symmetric system as for instance a multi-layer material (Fig. 2). Through the quadrupole technique, we obtain:

$$\begin{bmatrix} \theta_i \\ \phi_i \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} \theta_o \\ \phi_o \end{bmatrix}$$

for the bi-layer material (6)

and

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Nomenclature

a	thermal diffusivity	$m^2\,s^{-1}$
C_p	specific heat	$J\,kg^{-1}\,K^{-1}$
e	thickness	m
G	Green's function	
h	heat exchange coefficient	$W\,m^{-2}\,K^{-1}$
p	Laplace variable	s^{-1}
q, q'	heat flux density	$W\,m^{-2}$
S, S'	cross-section	m^2
T	temperature	K

Greek symbols

δ	Dirac function	
ϕ	Laplace transformed heat flux	W s
λ	thermal conductivity	$W\,m^{-1}\,K^{-1}$
ρ	density	$kg\,m^{-3}$
θ	Laplace transformed temperature	K s

Subscripts

i	input
o	output

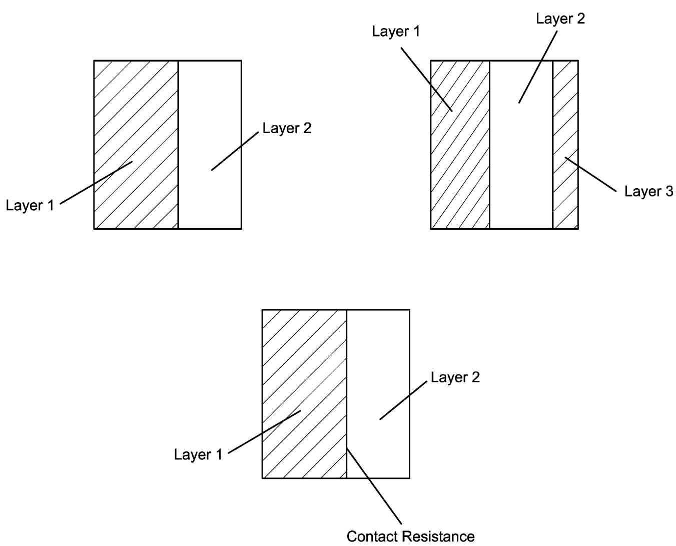

Fig. 2. Non-symmetric multi-layer.

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}A_3 & B_3\\ C_3 & D_3\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix}$$

for the tri-layer material (7)

and

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}1 & R_c\\ 0 & 1\end{bmatrix}\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix}$$

for the bi-layer material with contact resistance (8)

or

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix} \tag{9}$$

and

$$\theta_o=\frac{1}{\mathbf{C}}\phi_i \tag{10}$$

with:

$$\mathbf{C}_{12}=C_1A_2+D_1C_2 \tag{11}$$

and

$$\mathbf{C}_{123}=(C_1A_2+D_1C_2)A_3+(C_1B_2+D_1D_2)C_3 \tag{12}$$

and

$$\mathbf{C}_{1R2}=C_1A_2+(C_1R_c+D_1)C_2 \tag{13}$$

Changing the inlet with the outlet:

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix}$$

for the bi-layer material (14)

and

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}A_3 & B_3\\ C_3 & D_3\end{bmatrix}\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix}$$

for the tri-layer material (15)

and

$$\begin{bmatrix}\theta_i\\ \phi_i\end{bmatrix}=\begin{bmatrix}A_2 & B_2\\ C_2 & D_2\end{bmatrix}\begin{bmatrix}1 & R_c\\ 0 & 1\end{bmatrix}\begin{bmatrix}A_1 & B_1\\ C_1 & D_1\end{bmatrix}\begin{bmatrix}\theta_o\\ \phi_o\end{bmatrix}$$

for the bi-layer material with contact resistance (16)

or

$$\mathbf{C}_{21}=A_1C_2+C_1D_2 \tag{17}$$

and

$$\mathbf{C}_{321}=(A_2C_3+C_2D_3)A_1+(B_2C_3+D_2D_3)C_1 \tag{18}$$

and

$$\mathbf{C}_{2R1}=A_1C_2+C_1(C_2R_c+D_2) \tag{19}$$

Using the property that $A_i=D_i$ (symmetry), we obtain:

$$\mathbf{C}_{12}=\mathbf{C}_{21} \tag{20}$$

and

$$\mathbf{C}_{123}=\mathbf{C}_{321} \tag{21}$$

and

$$\mathbf{C}_{1R2}=\mathbf{C}_{2R1} \tag{22}$$

Whatever is the number of layers, the result is the same. This can be easily shown from Eq. (9). The inverse of this relation is given by:

$$\begin{bmatrix}\theta_i\\ -\phi_i\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1}\begin{bmatrix}\theta_o\\ -\phi_o\end{bmatrix} \tag{23}$$

with $\phi_o=0$, we find:

$$-\phi_i=\frac{1}{\Delta}(-\mathbf{C})\theta_o \quad \text{or} \quad \theta_o=\frac{\mathbf{AD}-\mathbf{BC}}{\mathbf{C}}\phi_i \tag{24}$$

as $\Delta = \mathbf{AD} - \mathbf{BC} = 1$ (passive system), we obtain:

$$\theta_o = \frac{\phi_i}{\mathbf{C}} \tag{25}$$

(same expression as Eq. (10)).

2. Bi-layer with a non-plane interface

A study in steady-state regime [2] carried out on a bi-layer material with a non-plane interface exhibits large variations of the thermal resistance compared to the case of a plane interface. Experiments in transient regime have also been performed on a non-homogeneous sample as described in the Fig. 3 and composed of epoxy resin and brass.

2.1. Experimental results

In order to compare their respective behaviours, the sample shown in the Fig. 3 as well as the bi-layer material with a plane interface (Fig. 4) are made with the same quantities of materials. A flash experiment is considered. The sample is subjected to a heat flux stimulation with a very short duration (Dirac). The in-time temperature response is measured through a semiconductor thermocouple (Fig. 5).

Samples are tested with the stimulation both in the brass and in the resin sides. The four experimental thermograms are drawn in Fig. 6.

We can notice that for the sample with a plane interface, the responses are the same whatever is the stimulated face (consistent with theory). On the other hand, as for the study in steady-state regime, the responses are faster for the sample with a non-plane interface than for the plane interface. In addition to that, some differences appear according to the stimulated face, what let us suppose the presence of a “thermal diode” effect. The following theoretical study will show that it is not the case in reality.

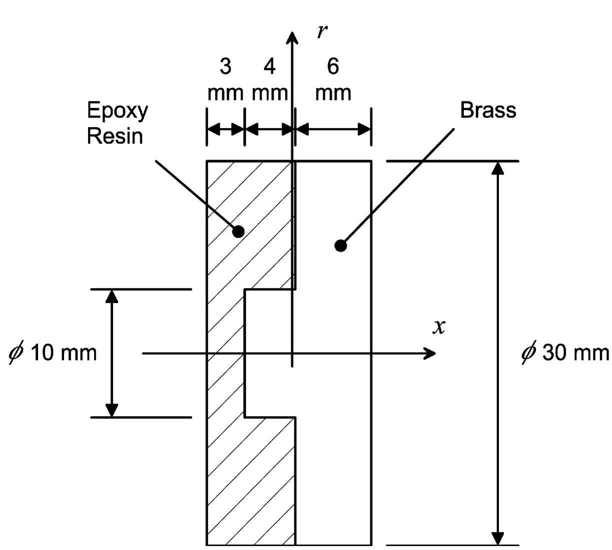

Fig. 3. Bi-layer with a non-plane interface.

2.2. Numerical study

No analytical solution exists for this problem. So, it has been solved numerically through the finite elements numerical code FlexPDE. Computations were performed in a 3D axi-symmetric geometry assuming that there are no contact resistance between the two materials and no heat loss with the surrounding, what is not really the case in our experiment where heat exchange by convection takes place.

Nominal values used for simulation:

– Epoxy resin:

$\lambda = 0.19\ \mathrm{W\,m^{-1}\,K^{-1}}$
$\rho = 1180\ \mathrm{kg\,m^{-3}}$
$C_p = 1440\ \mathrm{J\,kg^{-1}\,K^{-1}}$

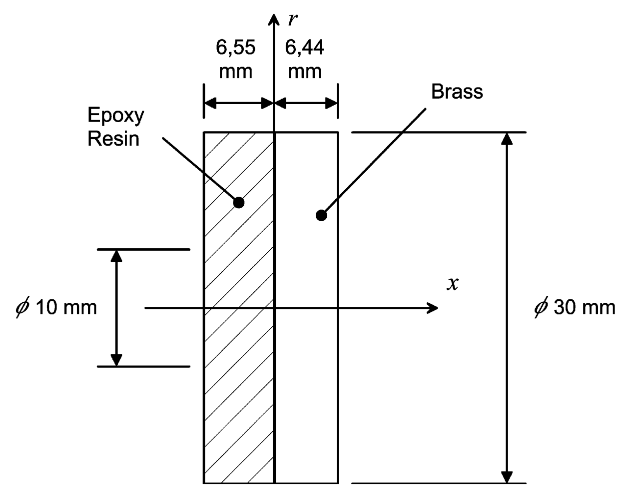

Fig. 4. Bi-layer with a plane interface.

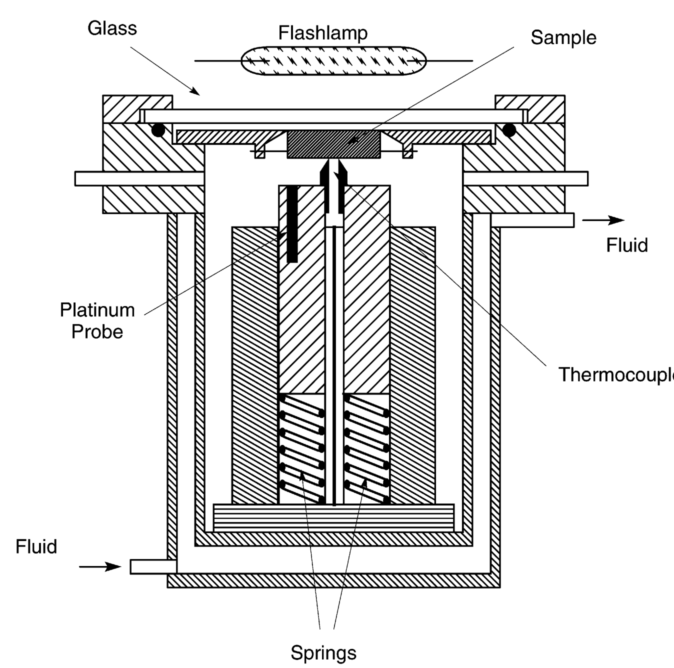

Fig. 5. Flash type diffusivimeter.

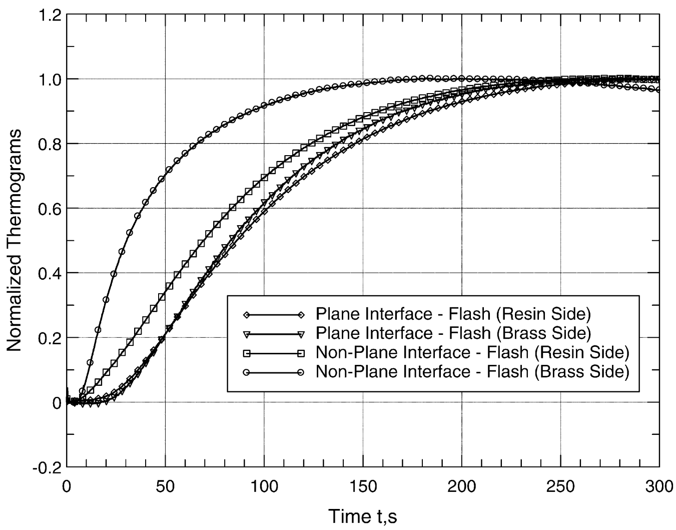

Fig. 6. Experimental curves.

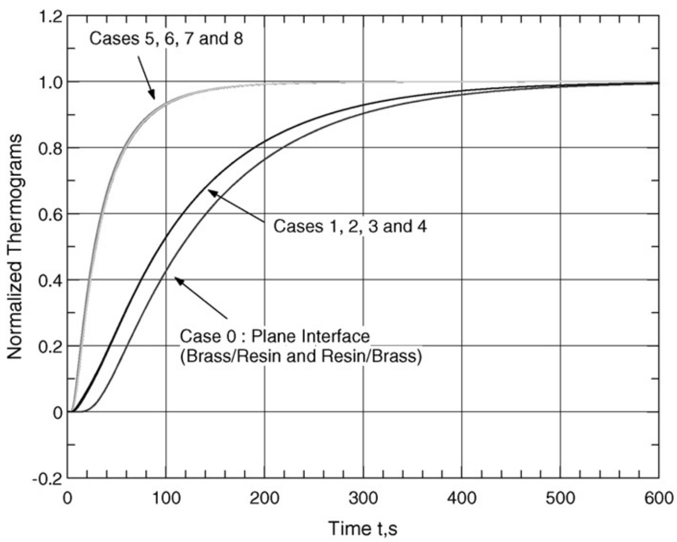

Fig. 7. Theoretical thermograms.

– Brass:

$$\lambda = 115 \text{ W m}^{-1}\text{ K}^{-1}$$
$$\rho = 8500 \text{ kg m}^{-3}$$
$$C_p = 385 \text{ J kg}^{-1}\text{ K}^{-1}$$

2.2.1. *First case*

At first, we considered a uniform stimulation on the front face and a mean temperature measurement on the back face (we called it "uniform"). In that case and after permutation of the stimulation and the detection (Fig. 8, cases #1 and #2), thermograms overlap (see Fig. 7). Thus, contrary to the experiment, no diode effect appears in simulations.

2.2.2. *Second case*

In experiments, temperature measurement by thermocouple is local. So, we are now interested in the case of an uniform stimulation and a mean temperature measurement on a restricted area corresponding to 1/3 of the sample diameter and centered on the sample axis (we called it "local"). In that case (Fig. 8, case #6), the response is different from that obtained in the case #2. However, if we change the stimulation with the detection (local stimulation and uniform measurement case #5), the same thermogram is obtained.

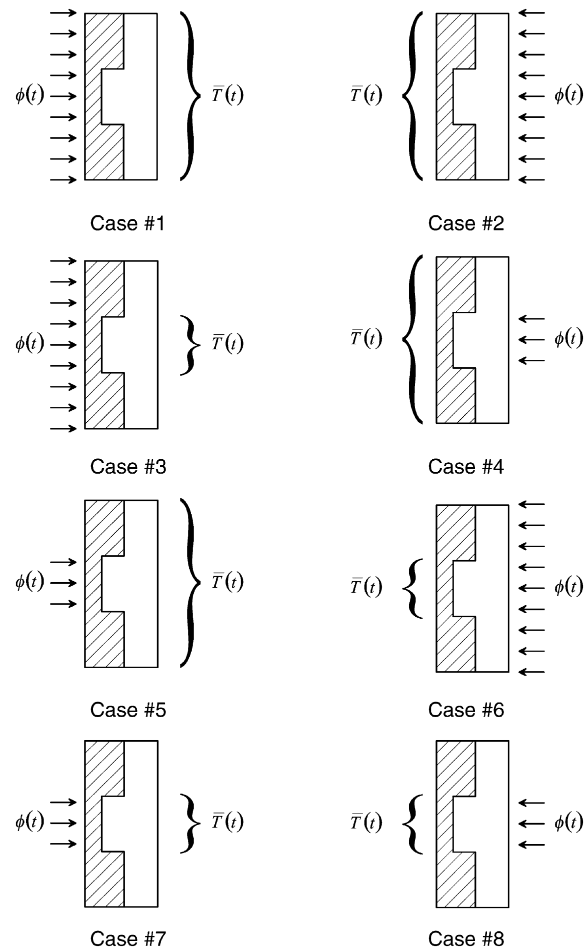

Fig. 8. Various configurations.

2.2.3. *General case*

If we consider the 8 possible cases (Fig. 8) (the Case 0 corresponds to the plane interface), we obtain two different responses that exhibit 3D effects in the sample. In all the cases, the responses are the same if the inlet and the outlet are swapped. Thus, there are no diode effect. The configurations related to the experiments correspond to the cases #3 and #6. The experimental results are now in good agreement with theory.

2.3. *Generalization of the results*

The question is then to know if these results can be extended to any other conductive systems. In fact, as it will be shown below, the answer is yes.

Let us consider now the heterogeneous system described in Fig. 9 without internal source and initially at thermal equilibrium. This system is general and concerns not only the multilayers system, but also all other systems that exhibit distributed phases as for instance resin materials containing heteroge-

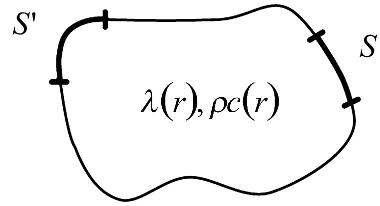

Fig. 9. Heterogeneous conductive system.

neously distributed particles or loads (balls, fibers, ...). Using the Green's functions (see Appendix A) with $g \equiv 0$ and $F \equiv 0$, the result is:

$$T(r,t) = \int_{\tau=0}^{\tau=t} \mathrm{d}\tau \int_S G(r,t|r',\tau) f(r',\tau)\,\mathrm{d}r' \tag{26}$$

If a Dirac of flux is imposed on a part S' of the surface with $f(r',\tau)|_{S'} = q'\delta(\tau)$ ($q' =$ constant), the mean temperature $\bar{T}$ on another part S is given by:

$$\begin{aligned}\bar{T}(t) &= \frac{1}{S}\int_S T(r,t)\,\mathrm{d}r = \frac{q'}{S}\int_S\int_{S'} G(r,t|r',0)\,\mathrm{d}r'\,\mathrm{d}r \\ &= \frac{q'}{S}\int_S\int_{S'} K(r,t|r')\,\mathrm{d}r'\,\mathrm{d}r\end{aligned} \tag{27}$$

Setting $K(r,t-\tau|r') = G(r,t|r',\tau)$ (in fact Green's function only depends on $t-\tau$ and not independently on t or τ).

However, if a Dirac of flux is now imposed on a part S of the surface with $f(r,\tau)|_S = q\delta(\tau)$ ($q =$ constant), the mean surface temperature $\bar{T}'$ on another part S' is given by:

$$\begin{aligned}\bar{T}'(t) &= \frac{1}{S'}\int_{S'} T(r',t)\,\mathrm{d}r' = \frac{q}{S'}\int_{S'}\int_S G(r',t|r,0)\,\mathrm{d}r\,\mathrm{d}r' \\ &= \frac{q}{S'}\int_{S'}\int_S K(r',t|r)\,\mathrm{d}r\,\mathrm{d}r'\end{aligned} \tag{28}$$

The order of integration on both surfaces that can be permuted and the reciprocity of G lead to:

$$K(r,t|r') = G(r,t|r',0) = G(r',0|r,-t) = K(r',t|r) \tag{29}$$

Finally, it comes:

$$\frac{S\bar{T}(t)}{q'} = \frac{S'\bar{T}'(t)}{q} \tag{30}$$

This shows the system reversibility.

3. Conclusion

A non-plane interface decreases the thermal resistance of the system and increases heat transfers (apparent diffusivity is larger than for a smooth interface). On the other hand, we showed that no thermal diode (in pure conduction) can exist either in steady-state or in transient regime.

The system being linear (λ and ρc non-dependent of temperature T), the results obtained for a Dirac of flux can be extended to any shape of stimulation. In fact, we have shown that the transfer function of the system is "reversible".

Appendix A. Green's functions for heat transfer equation in a non-homogeneous material

A.1. Green's function reciprocity (see [3])

Green's function $G(r,t|a,\tau)$ satisfies:

$$\nabla\cdot[\lambda\nabla G(r,t|a,\tau)] + \delta(r-a)\delta(t-\tau) = \rho c\frac{\partial G}{\partial t} \quad \text{in } R,\ t>\tau \tag{A.1}$$

$$\lambda\frac{\partial G}{\partial n} + hG = 0 \quad \text{on } S,\ t>\tau \tag{A.2}$$

$$G(r,t|a,\tau) = 0 \quad \text{if } t<\tau \tag{A.3}$$

In this equation λ, ρc and h can vary in space (different materials). We have also to notice that, defined in this way, the Green's function unit is $\mathrm{K\,J^{-1}}$.

The reciprocity consists in showing that:

$$G(b,\theta|a,\tau) = G(a,-\tau|b,-\theta) \tag{A.4}$$

A.1.1. Demonstration

Let:

$$F(r,t) = G(r,t|a,\tau) \tag{A.5}$$

$$H(r,t) = G(r,-t|b,-\theta) \tag{A.6}$$

F and H satisfy the following equations:

$$\nabla\cdot[\lambda\nabla F] + \delta(r-a)\delta(t-\tau) = \rho c\frac{\partial F}{\partial t} \quad \text{in } R,\ t>\tau \tag{A.7}$$

$$\lambda\frac{\partial F}{\partial n} + hF = 0 \quad \text{on } S,\ t>\tau \tag{A.8}$$

$$F(r,t) = 0 \quad \text{if } t<\tau \tag{A.9}$$

$$\nabla\cdot[\lambda\nabla H] + \delta(r-b)\delta(t-\theta) = -\rho c\frac{\partial H}{\partial t} \quad \text{in } R,\ t>\tau \tag{A.10}$$

$$\lambda\frac{\partial H}{\partial n} + hH = 0 \quad \text{on } S,\ t>\tau \tag{A.11}$$

$$H(r,t) = 0 \quad \text{if } t>\theta \tag{A.12}$$

Multiplying Eq. (A.7) by H, Eq. (A.10) by F and taking the difference, we finally obtain after an integration in space r on R and in time t for $-\infty < t < +\infty$:

$$\int_{t=-\infty}^{t=+\infty}\int_R \big[H\nabla\cdot(\lambda\nabla F) - F\nabla\cdot(\lambda\nabla H)\big]\,\mathrm{d}r\,\mathrm{d}t + H(a,\tau) - F(b,\tau) = \int_R \rho c[FH]_{t=-\infty}^{t=+\infty}\,\mathrm{d}r \tag{A.13}$$

The first left-hand side term is null. Indeed (Green's formula):

$$\begin{aligned}&\int_R \big[H\nabla\cdot(\lambda\nabla F) - F\nabla\cdot(\lambda\nabla H)\big]\,\mathrm{d}r \\ &\quad= \int_R \big[\nabla\cdot(H\lambda\nabla F) - \nabla\cdot(F\lambda\nabla H)\big]\,\mathrm{d}r \\ &\quad= \int_S \lambda\left(H\frac{\partial F}{\partial n} - F\frac{\partial H}{\partial n}\right)\mathrm{d}S = 0\end{aligned} \tag{A.14}$$

The nullity of this term comes from the homogeneous character of the boundary condition (A.2).

Finally, the term $[FH]_{t=-\infty}^{t=+\infty}$ is equal to zero because of the initial condition (Green's function is null before the stimulation) for F at $t=-\infty$ (A.9) and for H at $t=+\infty$ (A.12). It comes:

$$F(b,\theta)=H(a,\tau) \tag{A.15}$$

$$G(b,\theta|a,\tau)=G(a,-\tau|b,-\theta) \tag{A.16}$$

A.2. Using the Green's function

To obtain $T(r,t)$, the following system must be solved:

$$\begin{cases} \nabla\cdot\left[\lambda\nabla T(r,t)\right]+g(r,t)=\rho c\dfrac{\partial T(r,t)}{\partial t} & \text{in } R,\ t>0 \\ \lambda\dfrac{\partial T}{\partial n}+hG=f(r,t) & \text{on } S,\ t>0 \\ T(r,t)=F(r) & \text{at } t=0 \end{cases} \tag{A.17}$$

Let use the Green's function $G(r,t|r',\tau)$, which satisfies the relation of reciprocity (16):

$$G(r,t|r',\tau)=G(r',-\tau|r,-t) \tag{A.18}$$

r and t being fixed, it satisfies the following problem in r' and τ (∇_0 denotes the operator related to the variable r'):

$$\begin{cases} \nabla_0\cdot[\lambda\nabla_0 G]+\delta(r'-r)\delta(\tau-t) \\ \quad =-\rho c\dfrac{\partial G(r',\tau)}{\partial\tau} & \text{in } R,\ t>\tau \\ \lambda\dfrac{\partial G}{\partial n}+hG=0 & \text{on } S,\ t>\tau \\ G=0 & \text{if } t<\tau \end{cases} \tag{A.19}$$

After a change of variables (r' and τ), T satisfies:

$$\begin{cases} \nabla_0\cdot\left[\lambda\nabla_0 T(r',\tau)\right]+g(r',\tau) \\ \quad =\rho c\dfrac{\partial T(r',\tau)}{\partial\tau} & \text{in } R,\ \tau>0 \\ \lambda\dfrac{\partial T}{\partial n}+hT=f(r',\tau) & \text{on } S,\ \tau>0 \\ T(r',\tau)=F(r') & \text{if } t=0 \end{cases} \tag{A.20}$$

Multiplying Eq. (A.19.a) by T, Eq. (A.20.a) by G and taking the difference yields:

$$\begin{aligned} &\left[G\nabla_0\cdot(\lambda\nabla_0 T)-T\nabla_0\cdot(\lambda\nabla_0 G)\right]+g(r',\tau)G \\ &\quad-\delta(r'-r)\delta(\tau-t)T(r',\tau)=\rho c\frac{\partial(GT)}{\partial\tau} \end{aligned} \tag{A.21}$$

Integrating the previous equation in r' on all the volume R and in τ for $0<\tau<t^*=t+\varepsilon$ (ε being arbitrarily small):

$$\begin{aligned} &\int_{\tau=0}^{\tau=t^*}\mathrm{d}\tau\int_R\left[G\nabla_0\cdot(\lambda\nabla_0 T)-T\nabla_0\cdot(\lambda\nabla_0 G)\right]\mathrm{d}r' \\ &\quad+\int_{\tau=0}^{\tau=t^*}\mathrm{d}\tau\int_R g(r',\tau)G\,\mathrm{d}r'-T(r,t)=\int_R\rho c[GT]_{\tau=0}^{\tau=t^*}\,\mathrm{d}r' \end{aligned} \tag{A.22}$$

Let us estimate successively the different terms appearing in the previous equation. Using the Green's formula and the boundary conditions on S, we obtain:

$$\begin{aligned} &\int_R\left[G\nabla_0\cdot(\lambda\nabla_0 T)-T\nabla_0\cdot(\lambda\nabla_0 G)\right]\mathrm{d}r' \\ &\quad=\int_S\lambda\left(G\frac{\partial T}{\partial n}-T\frac{\partial G}{\partial n}\right)\mathrm{d}r' \\ &\quad=\int_S f(r',\tau)G\,\mathrm{d}r' \end{aligned} \tag{A.23}$$

$$[GT]_{\tau=0}^{\tau=t^*}=(GT)|_{\tau=t^*}-(GT)|_{\tau=0}=-G|_{\tau=0}F(r') \tag{A.24}$$

Because $G|_{\tau=t^*}=G(r',-(t+\varepsilon)|r,-t)=0$. It finally comes:

$$\begin{aligned} T(r,t)=&\int_{\tau=0}^{\tau=t^*}\mathrm{d}\tau\int_S G(r,t|r',\tau)f(r',\tau)\,\mathrm{d}r' \\ &+\int_{\tau=0}^{\tau=t^*}\mathrm{d}\tau\int_R G(r,t|r',\tau)g(r',\tau)\,\mathrm{d}r' \\ &+\int_R\rho cG(r,t|r',\tau)|_{\tau=0}F(r')\,\mathrm{d}r' \end{aligned} \tag{A.25}$$

In the particular case with no internal sources ($g\equiv 0$) and a uniform initial condition ($F\equiv 0$), the result is:

$$T(r,t)=\int_{\tau=0}^{\tau=t}\mathrm{d}\tau\int_S G(r,t|r',\tau)f(r',\tau)\,\mathrm{d}r' \tag{A.26}$$

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